# ON PERIODIC SOLUTIONS OF BURGERS' EQUATION

(O PERIODICHESKIKH RESHENIIAKH URAVNENIIA BIURGERSA)

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N. N. KOCHINA
(Moscow)

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The investigation of periodic solutions of the equations of motion of a viscous fluid is of much interest. Such solutions of the Oseen approximation have been studied in the work of Lin [6].

The differential equation

$$\frac{\partial v}{\partial \tau} + v\frac{\partial v}{\partial x} = \mu\frac{\partial^2 v}{\partial x^2} \tag{0.1}$$

was first introduced by Burgers as a simple model for the equations of motion of a viscous fluid. In [1] some particular solutions of this equation are found. In [2,3] the general properties of Equation (0.1) are investigated, and it is shown that with the help of a certain substitution it can be reduced to the equation of heat conduction; also, the proof is given for the existence and uniqueness of the solution of the problem with given initial conditions.

Below we examine the solutions of the nonlinear partial differential equation (0.1), periodic with respect to time (period $T$).

**1.** Equation (0.1) may be put in the form

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} \qquad \left(\tau = \frac{T}{2\pi}t,\ v = \sqrt{\frac{2\pi\mu}{T}}w,\ x = \sqrt{\frac{\mu T}{2\pi}}y\right) \tag{1.1}$$

The problem is posed in the following form: to find a periodic solution of Equation (1.1) in the upper half-plane $y > 0$, when the values of the function $w$ are given on the axis $y = 0$ in the form of a periodic function of time, and the value of $w$ at infinity is equal to some non-positive constant

$$w(0,\ t) = \psi(t), \qquad \lim_{y\to\infty} w(y,\ t) = w_\infty \leqslant 0 \tag{1.2}$$

We shall assume that $\psi(t)$ is continuous and can be developed in a

Fourier series with coefficients of the order $1/k^r (r \geqslant 2)$

$$\psi(t) = \frac{U_0}{2} + \sum_{k=1}^{\infty} (U_k \cos kt + V_k \sin kt) \tag{1.3}$$

With the help of the substitution

$$w = -\frac{2}{u}\frac{\partial u}{\partial y} \tag{1.4}$$

Equation (1.1) is reduced to the equation of heat conduction

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \tag{1.5}$$

Thus, we have to look for a solution of the equation of heat conduction, satisfying the condition

$$-2\frac{\partial u}{\partial y}(0, t) = u(0, t)\psi(t) \tag{1.6}$$

such that the function $w(y, t)$ defined by (1.4) is periodic with period $2\pi$. We require for definiteness that the function $u(y, t)$ for all $y \geqslant 0$ be essentially positive.

From the condition of periodicity of the function $w(y, t)$, and making use of the relations (1.4) and (1.5), we obtain for the function $u(y, t)$ the functional relation ($\zeta$ is a real constant)

$$u(y, t + 2\pi) = \exp(-2\pi\zeta)\, u(y, t) \tag{1.7}$$

The solution of the functional relation (1.7) has the form [4]

$$u(y, t) = \exp(-\zeta t)\, \omega(y, t) \tag{1.8}$$

Here $\omega(y, t)$ is a periodic function of time with period $2\pi$.

**2.** We write out the solution of the heat-conduction equation (1.5) in the form of (1.8)

$$u(y, t) = \exp(-\zeta t)\{f(y, \zeta) +$$

$$+ \sum_{k=1}^{\infty} \exp \rho_k(\zeta)\, y\, [A_k \cos(kt - \omega_k(\zeta)\, y) + B_k \sin(kt - \omega_k(\zeta)\, y)] \tag{2.1}$$

$$\rho_k(\zeta) = -\frac{k}{\sqrt{2(\sqrt{k^2+\zeta^2}+\zeta)}}, \qquad \omega_k(\zeta) = \sqrt{\frac{\sqrt{k^2+\zeta^2}+\zeta}{2}}$$

$$f(y, \zeta) = \begin{cases} \frac{1}{2} A_0 \cos\sqrt{\zeta}y + \frac{1}{2} B_0 \sin\sqrt{\zeta}y/\sqrt{\zeta} & \text{for } \zeta > 0 \\ \frac{1}{2} A_0 + \frac{1}{2} B_0 y & \text{for } \zeta = 0 \\ \frac{1}{2} A_0 \operatorname{ch}\sqrt{-\zeta}\, y + \frac{1}{2} B_0 \operatorname{sh}\sqrt{-\zeta}\, y / \sqrt{-\zeta} & \text{for } \zeta < 0 \end{cases} \tag{2.2}$$

From Equations (2.1) and (2.2) it is evident that the quantities $A_k (k = 0, 1, 2, \ldots)$ and $B_k (k = 1, 2, \ldots)$ are the coefficients of a Fourier-series development of the function $\omega(0, t) = u(0, t) \exp(\zeta t)$. We now introduce the quantities $a_k (k = 0, 1, 2, \ldots)$ and $b_k (k = 1, 2, \ldots)$, satisfying the equalities

$$a_0 = B_0, \quad A_k = -\sigma_k a_k + \tau_k b_k, \quad B_k = -\tau_k a_k - \sigma_k b_k \quad (k=1, 2, \ldots) \tag{2.3}$$

$$\sigma_k = \left(\frac{k^2}{2(k^2+\zeta^2)\,[\sqrt{k^2+\zeta^2}+\zeta]}\right)^{1/2}, \qquad \tau_k = \left(\frac{\sqrt{k^2+\zeta^2}+\zeta}{2(k^2+\zeta^2)}\right)^{1/2} \tag{2.4}$$

Differentiating (2.1) with respect to $y$, with the notation of (2.2) and (2.4), we obtain

$$\frac{\partial u(y, t)}{\partial y} = \exp(-\zeta t)\left\{\frac{\partial f(y, \zeta)}{\partial y} + \sum_{k=1}^{\infty} \exp \rho_k(\zeta)\, y\, [a_k \cos(kt - \omega_k(\zeta)\, y) + b_k \sin(kt - \omega_k(\zeta)\, y)]\right\} \tag{2.5}$$

Equations (2.5) and (2.2) show that $a_k (k = 0, 1, 2, \ldots)$ and $b_k (k = 1, 2, \ldots)$ are Fourier coefficients of the function $\partial\omega(0, t)/\partial y$.

It is now possible to write out the expression for the function $w(y, t)$ in an obvious form, making use of Equations (1.4), (2.1), and (2.5):

$$w(y, t) = -2\,\frac{\dfrac{\partial f}{\partial y} + \sum_{k=1}^{\infty} \exp \rho_k(\zeta)\, y\,[a_k \cos(kt - \omega_k(\zeta)\, y) + b_k \sin(kt - \omega_k(\zeta)\, y)]}{f + \sum_{k=1}^{\infty} \exp \rho_k(\zeta)\, y\,[A_k \cos(kt - \omega_k(\zeta)\, y) + B_k \sin(kt - \omega_k(\zeta)\, y)]} \tag{2.6}$$

We now show, taking off from Equation (2.6), that condition (1.2) at infinity can be satisfied. Letting $y$ go to infinity, we have

$$\lim_{y\to\infty} w(y, t) = -2 \lim_{y\to\infty} \frac{\partial \ln f(y, \zeta)}{\partial y} \tag{2.7}$$

where

$$-\frac{2\partial \ln f(y, \zeta)}{\partial y} = \begin{cases} 2\sqrt{\zeta}\, \mathrm{tg}\,\{\sqrt{\zeta}\, y - \mathrm{arcctg}\, \sqrt{\zeta}\,(A_0/B_0)\} & \textbf{for } \zeta > 0 \\ -2B_0/(A_0 + B_0 y) & \textbf{for } \zeta = 0 \\ -2\sqrt{-\zeta}\,\dfrac{A_0\sqrt{-\zeta} + B_0 \,\mathrm{cth}\, \sqrt{-\zeta}\, y}{B_0 + A_0\sqrt{-\zeta}\,\mathrm{cth}\,\sqrt{-\zeta}\, y} & \textbf{for } \zeta < 0 \end{cases} \tag{2.8}$$

It is evident that Expression (2.8) gives a stationary solution of Equation (1.1) or, in other words, the solution $w$ of the equation

$$w\frac{dw}{dy} = \frac{d^2 w}{dy^2} \tag{2.9}$$

Letting $y$ go to infinity, in (2.8) we see that for $\zeta > 0$ a limit for $w(y, t)$ does not exist, and for $\zeta \leqslant 0$ this limit is equal to

$$w_\infty = -2\sqrt{-\zeta} \qquad (2.10)$$

Thus, for every given $w_\infty \leqslant 0$ the quantity $\zeta$ appearing in the solution (2.6) is determined from the following formula:

$$\zeta = -\tfrac{1}{4} w_\infty^2 \qquad (2.11)$$

It remains to determine the constants $a_k$ and $b_k$. Equating the Fourier coefficients of the left- and right-hand sides of Expression (1.6) (making use here of Equations (2.1) and (2.5) for $y = 0$), we obtain, under the assumption that $A_0$ is arbitrary, an infinite system of linear equations for $a_k (k = 0, 1, 2, \ldots)$ and $b_k (k = 1, 2, \ldots)$ (from the form of $w(y, t)$ in (2.6) it is evident that all these constants are determined exactly, up to the multiplier $1/A_0$)

$$a_k = \sum_{n=1}^{\infty} (\alpha_n^{(k)} a_n + \beta_n^{(k)} b_n) - \tfrac{1}{4} A_0 U_k \qquad (k = 1, 2, 3, \ldots) \qquad (2.12)$$

$$b_k = \sum_{n=1}^{\infty} (\gamma_n^{(k)} a_n + \delta_n^{(k)} b_n) - \tfrac{1}{4} A_0 V_k \qquad (k = 1, 2, 3, \ldots) \qquad (2.13)$$

where the following notation has been introduced: (2.14)

$$\alpha_n^{(k)} = \begin{cases} {}^1/_4 \sigma_n (U_{n+k} + U_{k-n}) + {}^1/_4 \tau_n (V_{n+k} - V_{k-n}) & (n < k) \\ {}^1/_4 \sigma_n (U_{n+k} + U_{n-k}) + {}^1/_4 \tau_n (V_{n+k} + V_{n-k}) & (n \geqslant k) \end{cases}$$

$$\beta_n^{(k)} = \begin{cases} -{}^1/_4 \tau_n (U_{n+k} + U_{k-n}) + {}^1/_4 \sigma_n (V_{n+k} - V_{k-n}) & (n < k) \\ -{}^1/_4 \tau_n (U_{n+k} + U_{n-k}) + {}^1/_4 \sigma_n (V_{n+k} + V_{n-k}) & (n \geqslant k) \end{cases}$$

$$\gamma_n^{(k)} = \begin{cases} {}^1/_4 \sigma_n (V_{n+k} + V_{k-n}) - {}^1/_4 \tau_n (U_{n+k} - U_{k-n}) & (n < k) \\ {}^1/_4 \sigma_n (V_{n+k} - V_{n-k}) - {}^1/_4 \tau_n (U_{n+k} - U_{n-k}) & (n \geqslant k) \end{cases}$$

$$\delta_n^{(k)} = \begin{cases} -{}^1/_4 \tau_n (V_{n+k} + V_{k-n}) - {}^1/_4 \sigma_n (U_{n+k} - U_{k-n}) & (n < k) \\ -{}^1/_4 \tau_n (V_{n+k} - V_{n-k}) - {}^1/_4 \sigma_n (U_{n+k} - U_{n-k}) & (n \geqslant k) \end{cases}$$

Then, when the constants $a_k$, $b_k (k = 1, 2, \ldots)$ have been found, the constant $a_0$ is determined from the formula

$$a_0 = -\frac{A_0 U_0}{4} - \frac{1}{2} \sum_{k=1}^{\infty} \{a_k [-\sigma_k U_k - \tau_k V_k] + b_k [\tau_k U_k - \sigma_k V_k]\} \qquad (2.15)$$

The infinite system of linear equations, (2.12) to (2.13), will be completely regular [5] if, for all $k = 1, 2, \ldots$

$$\sum_{n=1}^{\infty}\{|\alpha_n^{(k)}|+|\beta_n^{(k)}|\}\leqslant 1-\theta,\ \sum_{n=1}^{\infty}\{|\gamma_n^{(k)}|+|\delta_n^{(k)}|\}\leqslant 1-\theta \qquad (0<\theta<1) \tag{2.16}$$

From Equation (2.4) it is easy to see that for all $k$ the inequality

$$|\tau_k|+|\sigma_k|=\tau_k+\sigma_k\leqslant\sqrt{2/k} \tag{2.17}$$

holds. In view of (2.14) and (2.17), the condition (2.17) is satisfied if the series

$$\frac{|U_0|}{2}+\sum_{k=1}^{\infty}|U_k|=U, \qquad \sum_{k=1}^{\infty}|V_k|=V \tag{2.18}$$

converge, so that the inequality

$$U+V\leqslant\sqrt{2}\,(1-\theta) \tag{2.19}$$

is satisfied.

We shall assume that condition (2.19) is satisfied for the system of equations (2.12) to (2.13). Then the system (2.12) to (2.13) will be completely regular, and in order that it have a unique bounded solution [5] (the principal solution, i.e. the solution found by the method of successive approximations for initial conditions which are zero or arbitrary but bounded in total) it is sufficient that the quantities $-A_0U_k/4$, $-A_0V_k/4$ ($k=1, 2, 3, \ldots$) be bounded. But in view of Equation (1.3) these quantities will be Fourier coefficients of the function $-A_0\psi(t)/4$, from which their boundedness follows.

Then with the fulfillment of condition (2.19) it is possible to find uniquely from the system (2.12) to (2.13) the system of quantities $a_k$, $b_k$ which are bounded by a certain constant $K$. It remains to show that these quantities can be Fourier coefficients of some function and to show under what conditions the function $u(y, t)$ for all $y\geqslant 0$ and $t\geqslant 0$ remains positive.

**3.** We shall solve the system of equations (2.12) to (2.13) by the method of successive approximations. We note that the $m$th approximation $a_k^{(m)}$, $b_k^{(m)}$ is found by putting the $m-1$th approximation in the right-hand side of Equations (2.12) to (2.13):

$$a_k^{(m)}=\sum_{n=1}^{\infty}(\alpha_n^{(k)}a_n^{(m-1)}+\beta_n^{(k)}b_n^{(m-1)})-\frac{1}{4}A_0U_k \qquad (k=1,2,\ldots) \tag{3.1}$$

$$b_k^{(m)}=\sum_{n=1}^{\infty}(\gamma_n^{(k)}a_n^{(m-1)}+\delta_n^{(k)}b_n^{(m-1)})-\frac{1}{4}A_0V_k \qquad (k=1,2,\ldots) \tag{3.2}$$

For the zeroth approximation we take

$$a_k^{(0)} = b_k^{(0)} = 0 \qquad (k = 1, 2, \ldots) \tag{3.3}$$

For the first approximation we find from Equations (3.1), (3.2) and (3.3)

$$a_k^{(1)} = -\frac{1}{4}A_0 U_k, \qquad b_k^{(1)} = -\frac{1}{4}A_0 V_k \qquad (k = 1, 2, \ldots) \tag{3.4}$$

We shall consider that the Fourier coefficients of the function $\psi(t)$ satisfy the relations (with $e$ and $f$ constant)

$$|U_n| \leqslant e / n^r, \qquad |V_n| \leqslant f / n^r \qquad (r \geqslant 2) \tag{3.5}$$

Making use of Equations (2.4), (2.14), (3.1) to (3.5), it may be shown that if the condition

$$(1 + 2^r)(E + F) + (2 + 2^r)(U + V) < 2 \left( E = e \sum_{n=1}^{\infty} \frac{1}{n^r}, \; F = f \sum_{n=1}^{\infty} \frac{1}{n^r} \right) \tag{3.6}$$

is fulfilled, then the coefficients $a_n$ and $b_n$ satisfy the inequalities

$$|a_n| \leqslant \frac{1}{4}A_0 |U_n| + \Sigma, \qquad |b_n| \leqslant \frac{1}{4}A_0 |V_n| + \Sigma$$
$$\Sigma = \frac{A_0}{8} \frac{[(1 + 2^r)(E + F) + (2 + 2^r)(U + V)]\,[e + f]}{\{2 - (1 + 2^r)(E + F) - (2 + 2^r)(U + V)\}\, n^r} \tag{3.7}$$

and therefore can be Fourier coefficients of some function.

**4.** We shall now show under what conditions the function $u(y, t)$ will be positive for all non-negative values of $y$ and $t$. From Equations (2.1) and (2.2) it is evident that $u(y, t) > 0$, if for all $y \geqslant 0$

$$f(y, \zeta) > \sum_{k=1}^{\infty} [|A_k| + |B_k|] \tag{4.1}$$

making use of (2.2) it is possible to convince oneself that for $\zeta > 0$ the inequality (4.1) cannot be fulfilled, that for $\zeta = 0$ it holds if $A_0 > 0$, $B_0 \geqslant 0$

$$\frac{A_0}{2} > \sum_{k=1}^{\infty} [|A_k| + |B_k|] \tag{4.2}$$

and for $\zeta < 0$ it holds if $A_0 > 0$ and, either $B_0 \geqslant 0$ and (4.2) is satisfied, or the inequalities

$$B_0 < 0, \qquad \frac{A_0}{2}\left[1 + \frac{1}{\zeta}\left(\frac{B_0}{A_0}\right)^2\right] > \sum_{k=1}^{\infty} [|A_k| + |B_k|] \tag{4.3}$$

hold.

It is clear that if $B_0 \geqslant 0$, then since $u(0, t) > 0$ a given function

$\psi(t)$ must be subject to the condition

$$\int_0^{2\pi} \psi(t)\, u(0,\ t)\, dt \leqslant 0 \tag{4.4}$$

Thus, in this case a periodic function $\psi(t)$ must be either negative or change sign in the interval $0 \leqslant t \leqslant 2\pi$ at least once in such a way that the inequality (4.4) is fulfilled.

It is easy to see that if the term $c\,\partial w/\partial y$ is added to the left-hand side of Equation (1.1) ($c$ is an arbitrary constant), then the solution $w + c$ of this equation with the conditions (1.2), with $w_\infty \leqslant -c$, has the form of (2.6), where $\zeta = -(c + w_\infty)^2/4$, and $U_0$ in (2.12) to (2.15) must be changed to $U_0 + 2c$.

5. Let us now investigate the inverse problem: to find the solution $w(y,\ t)$ of Equation (1.1) when the function

$$\varphi(t) = \exp(\zeta t)\, \frac{\partial u(0,\ t)}{\partial y} \tag{5.1}$$

is given on $0 \leqslant t \leqslant 2\pi$, where $u(y,\ t)$ satisfies the heat-conduction equation (1.5), and, in addition, the value of the function $w(y,\ t)$ at infinity is given. Relations (5.1) and (1.8) show that $\psi(t) = \partial\omega(0,t)/\partial y$, and the constant $\zeta$ is determined from Equation (2.11).

From Equations (1.7) and (5.1) it is clear that the function $\phi(t)$ is periodic with period $2\pi$. Let us develop it in a Fourier series

$$\varphi(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \tag{5.2}$$

Making use of Equations (2.3) to (2.5), (2.1) and (1.4), we find $w(y,\ t)$, the solution of Equation (1.1). Here $u(y,\ t)$ conserves its sign if $\zeta \leqslant 0$. For all $y \geqslant 0$ and $t \geqslant 0$, $u(y,\ t > 0$ if the arbitrary constant $A_0 > 0$ and either $a_0 = B_0 \geqslant 0$, whereupon inequality (4.2) holds, or for $\zeta < 0$ condition (4.3) is satisfied. For all $y \geqslant 0$ and $t \geqslant 0$, $u(y,\ t) < 0$ if $A_0 < 0$ and either $a_0 = B_0 \leqslant 0$, whereupon

$$\frac{A_0}{2} < -\sum_{k=1}^{\infty} [|A_k| + |B_k|] \tag{5.3}$$

or $a_0 > 0$ and

$$\frac{A_0}{2}\left[1 - \frac{1}{\zeta}\left(\frac{a_0}{A_0}\right)^2\right] < -\sum_{k=1}^{\infty} [|A_k| + |B_k|] \tag{5.4}$$

The value $\psi(t)$ of the function $w(y,\ t)$ on the axis $y = 0$ is given by Equation (1.2). The coefficients of the expansion of $\psi(t)$ in a Fourier series (1.3) can be found by equating Fourier coefficients of the left-and right-hand sides of Equation (1.6). Here, as was indicated above, we consider the constant $A_0$ to be arbitrary. For finding the quantities $U_k$, $V_k$ ($k$ = 0, 1, 2, ...) we obtain the infinite system of linear equations

$$U_k = \sum_{n=0}^{\infty} (\lambda_n^{(k)} U_n + \mu_n^{(k)} V_n) - \frac{4}{A_0} a_k \qquad (k = 0, 1, 2, \ldots) \tag{5.5}$$

$$V_k = \sum_{n=0}^{\infty} (\xi_n^{(k)} U_n + \eta_n^{(k)} V_n) - \frac{4}{A_0} b_k \qquad (k = 1, 2, \ldots) \tag{5.6}$$

Here

$$\lambda_n^{(k)} = \begin{cases} A_0^{-1} [\sigma_{k+n} a_{k+n} - \tau_{k+n} b_{k+n} + \sigma_{k-n} a_{k-n} - \tau_{k-n} b_{k-n}] & (n < k) \\ A_0^{-1} [\sigma_{k+n} a_{k+n} - \tau_{k+n} b_{k+n} + \sigma_{n-k} a_{n-k} - \tau_{n-k} b_{n-k}] & (n \geqslant k) \end{cases} \tag{5,7}$$

$$\mu_n^{(k)} = \begin{cases} A_0^{-1} [\tau_{k+n} a_{k+n} + \sigma_{k+n} b_{k+n} - \tau_{k-n} a_{k-n} - \sigma_{k-n} b_{k-n}] & (n < k) \\ A_0^{-1} [\tau_{k+n} a_{k+n} + \sigma_{k+n} b_{k+n} + \tau_{n-k} a_{n-k} + \sigma_{n-k} b_{n-k}] & (n \geqslant k) \end{cases}$$

$$\xi_n^{(k)} = \begin{cases} A_0^{-1} [\tau_{n+k} a_{n+k} + \sigma_{n+k} b_{n+k} + \tau_{k-n} a_{k-n} + \sigma_{k-n} b_{k-n}] & (n < k) \\ A_0^{-1} [\tau_{n+k} a_{n+k} + \sigma_{n+k} b_{n+k} - \tau_{n-k} a_{n-k} - \sigma_{n-k} b_{n-k}] & (n \geqslant k) \end{cases}$$

$$\eta_n^{(k)} = \begin{cases} A_0^{-1} [-\sigma_{n+k} a_{n+k} + \tau_{n+k} b_{n+k} + \sigma_{k-n} a_{k-n} - \tau_{k-n} b_{k-n}] & (n < k) \\ A_0^{-1} [-\sigma_{n+k} a_{n+k} + \tau_{n+k} b_{n+k} + \sigma_{n-k} a_{n-k} - \tau_{n-k} b_{n-k}] & (n \geqslant k) \end{cases}$$

Analogously to this, as was shown in Section 2, it can be shown that the system (5.5) to (5.6) will be completely regular if the series

$$\sum_{k=1}^{\infty} \frac{|a_k|}{\sqrt{k}} = C, \qquad \sum_{k=1}^{\infty} \frac{|b_k|}{\sqrt{k}} = D \tag{5.8}$$

converge, whereupon the inequality

$$C + D \leqslant \frac{|A_0|}{2\sqrt{2}} (1 - \theta) \qquad (0 < \theta < 1) \tag{5.9}$$

is fulfilled.

The quantities $-4a_k A_0^{-1}$ and $-4b_k A_0^{-1}$, being Fourier coefficients of some function, are bounded. It follows that the system (5.5) to (5.6) determines a unique bounded solution (principal solution). Analogously to this, as was done in Section 3, it is possible to show by solving the system of equations (5.5) to (5.6) by the method of successive approximations that the quantities $U_n$, $V_n$ can be Fourier coefficients of some function. For this we assume that

$$|a_n| \leqslant \frac{h}{n^r}, \qquad |b_n| \leqslant \frac{g}{n^r} \qquad (r \geqslant 2,\ n = 1, 2, \ldots) \tag{5.10}$$

If the condition

$$(1 + 2^r)(H + G) + (2 + 2^r)(A + B) < |A_0| / 2 \tag{5.11}$$

is fulfilled, where

$$A = \sum_{n=1}^{\infty} |a_n|, \quad B = \sum_{n=1}^{\infty} |b_n|, \quad H = h \sum_{n=1}^{\infty} \frac{1}{n^r}, \quad G = g \sum_{n=1}^{\infty} \frac{1}{n^r}$$

we obtain estimates for the coefficients of the series development (1.3) of the function $\psi(t)$

$$|U_n| \leqslant \frac{4}{|A_0|} |a_n| + \Delta; \qquad |V_n| \leqslant \frac{4}{|A_0|} |b_n| + \Delta$$

$$\Delta = \frac{1}{|A_0|} \left\{ \frac{[4 + (1 + 2^r) |U_0|] [(1 + 2^r)(H + G) + (2 + 2^r)(A + B)]}{|A_0| - 2[(1 + 2^r)(H + G) + (2 + 2^r)(A + B)]} + \right. \tag{5.12}$$
$$\left. + \frac{(1 + 2^r)|U_0|}{2} \right\} \frac{[h + g]}{n^r}$$

6. From Equations (3.7) and (5.12) it is clear that when conditions (3.6) and (5.11) are fulfilled the solution $a_k$, $b_k$ and $U_k$, $V_k$ of the infinite system of equations can be found. However, the majorant which we have investigated is rough. For certain specific problems it is possible to find much more accurate estimates. From this it is clear that in general it is possible to find the corresponding solutions, which need not necessarily satisfy conditions (3.6) and (5.11).

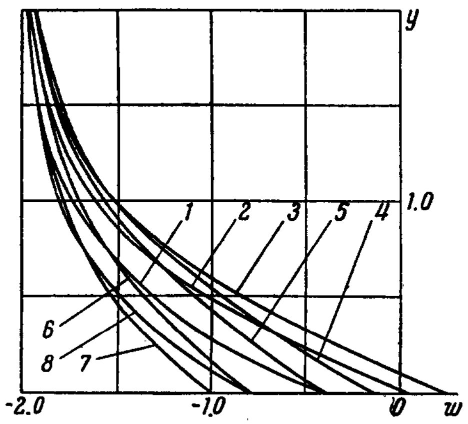


Fig. 1.

From Equations (1.1), (1.4) and (1.5), it is easy to see that condition (3.5), which is here assumed to be satisfied for the Fourier coefficients of $\psi(t)$, indicates continuity of all derivatives of the function $w(y, t)$ appearing in Equation (1.1) and, consequently, of the continuity of the function $u(y, t)$ and its derivatives up to the third, inclusive.

By way of illustration let us quote the results of some calculations. Figure 1 shows the curves for the relation $w = w(y)$ for the case where

$$\psi(t) = -0.6 + 0.4 \sin t \tag{6.1}$$

In Fig. 1 the curves *1* to *8* correspond to the values $t = 0, \pi/4, \pi/2, 7\pi/8, \pi, 5\pi/4, 3\pi/2, 7\pi/4$. The constant which determines the value of $w$ at infinity is $w_\infty = -2$, so that $\zeta = -1$ in accordance with (2.11).

Note that in this example $\partial w/\partial y < 0$ for all values of $t$ when $y \geqslant 0$; the derivative $\partial^2 w/\partial y^2$ changes sign at $y = 0$ in accordance with Equations (1.1), (2.6) and (6.1), though this is hardly noticeable.

In Fig. 2 are given the curves for the relation $w = w(y)$, when

$$\varphi(t) = \begin{cases} t & 0 \leqslant t \leqslant \frac{1}{2}\pi \\ \pi - t & \frac{1}{2}\pi \leqslant t \leqslant \frac{3}{2}\pi \\ t - 2\pi & \frac{1}{2}3\pi \leqslant t \leqslant 2\pi \end{cases} \tag{6.2}$$

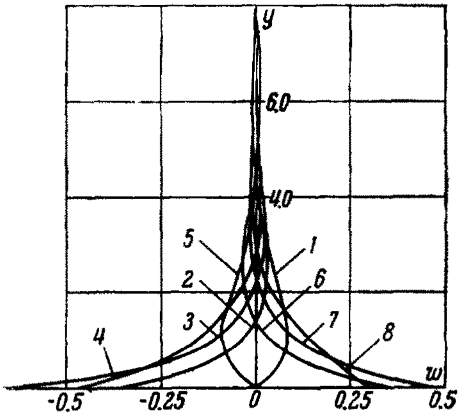


Fig. 2.

Two solutions (2.6) for $w(y)$ are drawn in Fig. 2 for values of the constant $A_0$ equal to 17.3307 and -17.3307. The curves correspond to pairs of values of $t$, namely:

1 $(0, \pi)$, 2 $(\pi/3, 4\pi/3)$, 3 $(\pi/2, 3\pi/2)$, 4 $(2\pi/3, 5\pi/3)$,
5 $(\pi, 0)$, 6 $(4\pi/3, \pi/3)$, 7 $(3\pi/2, \pi/2)$, 8 $(5\pi/3, 2\pi/3)$

corresponding to the first and second values of $A_0$.

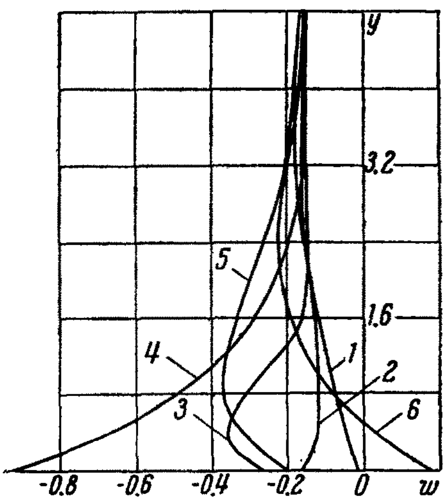


Fig. 3.

In Fig. 3 the relation $w = w(y)$ is given for the case

$$\varphi(t) = \begin{cases} t & (0 \leqslant t \leqslant \pi) \\ -2t + 3\pi & (\pi \leqslant t \leqslant \frac{5}{3}\pi) \\ t - 2\pi & (\frac{5}{3}\pi \leqslant t \leqslant 2\pi) \end{cases} \tag{6.3}$$

Here $A_0 = 17.3307$; the curves correspond to the values $t = 0$, $\pi/3$, $2\pi/3$, $\pi$, $4\pi/3$, $5\pi/3$.

Note that in the last two cases $w_\infty = 0$, so that $\zeta = 0$, in accordance with (2.11), and the function $u(y, t)$ in (1.4) will be a periodic solution of Equation (1.5).

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*Translated by* A.R.